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Non-static electromagnetic fields in general relativity: II

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Abstract. The Rainich equations of the 'already unified field theory' are studied in the case of non-static electromagnetic fields, and a solution is obtained for a space–time metric which admits a group G_4 of automorphisms. There exists a divergence-free electromagnetic field for $x^4 > 0$, except for $x^4 \to \infty$. It is shown that the electromagnetic field vanishes for large values of time, and the solution for a completely empty flat space is then obtained.

1. Introduction

Rainich (1925) has shown that Maxwell's equations reduce to a simple statement involving only the Ricci tensor. One then obtains from the standard theory of Maxwell and Einstein an 'already unified field theory' which is completely geometrical. Misner and Wheeler (1957) have recently contributed greatly to the subject and discussed in an elegant manner how the energy tensor can be used to describe the source-free electromagnetic field, and how this is readily interpreted in gravitational theory. In an otherwise empty Riemann space with appropriate signature the Rainich algebraic equations (Rainich 1925, Misner and Wheeler 1957) constitute a set of necessary and sufficient conditions to find the non-null electromagnetic field. Thus the Rainich equations of the 'already unified field theory', describing a physical situation in which gravitation and source-free electromagnetic fields may be present, are

$$R = 0 \tag{1}$$

$$R^{\mu}_{\nu}R^{\nu}_{\lambda} = \delta^{\mu}_{\lambda}(\tfrac{1}{4}R^{\beta}_{\alpha}R^{\alpha}_{\beta}) \tag{2}$$

$$R^{4}_{4} > 0 \tag{3}$$

$$\alpha_{\beta,\gamma} - \alpha_{\gamma,\beta} = 0 \tag{4}$$

where

$$\alpha_{\beta} = \frac{(-g)^{1/2}\epsilon_{\beta\lambda\mu\nu}R^{\lambda\gamma;\mu}R^{\nu}_{\gamma}}{R_{\sigma\delta}R^{\sigma\delta}} \tag{5}$$

g being the determinant of the metric tensor. The Levi-Civita symbol $\epsilon_{\beta\lambda\mu\nu}$ is skew symmetric in all pairs of indices, and ϵ_{4123} has the value unity.

Recently Witten (1959), Raychaudhuri (1960) and Datta (1961) have studied the solutions of the Rainich equations in the case of static fields. Later Rosen (1962) has obtained a spatially homogeneous solution of the Rainich equations in the case of non-static fields. Datta (1965) has lately studied the case of a non-static electromagnetic field for a space–time admitting a three-parameter Abelian group of motions, whose minimum invariant varieties are three-dimensional spaces, and presented solutions, one of which can, by a suitable transformation, assume the form of Rosen's solution. The aim of the present paper is to study the Rainich equations of the 'already unified field theory' in the case when the electromagnetic field is non-static and the space–time metric admitting a group G_4 of automorphisms has the structure

$$\begin{gathered}[X_i, X_j] = 0 \qquad (i, j = 1, 2, 3), \qquad [X_1, X_4] = lX_2 \\ [X_2, X_4] = mX_1, \qquad [X_3, X_4] = 0, \qquad l = m = 0, \pm 1.\end{gathered} \tag{6}$$

We note that a special non-static metric which admits an intransitive group of motions is considered, and that the group G_4 includes the Abelian subgroups G_3. It is well known that any type of group transformation, if it is intransitive, results in the space–time points being reflected on a point lying on some certain surface of transitiveness. The whole space–time thus divides into such surfaces and physically they represent invariant images.

2. The basic equations of the problem

We consider the case when the electromagnetic field is non-static and the space–time metric admitting a group G_4 of automorphisms has the structure (6). In this case the metric can be put in the form

$$ds^2 = (dx^4)^2 - \mu\{(dx^1)^2 + (dx^2)^2\} - \nu(dx^3)^2 \tag{7}$$

with

$$\mu \equiv \mu(x^4), \qquad \nu \equiv \nu(x^4).$$

Here x^4 is the time coordinate, x^1, x^2, x^3 are space coordinates and μ, ν are both positive. This metric admits an intransitive group of motions.

Now equation (1) is

$$R_1^1 + R_2^2 + R_3^3 + R_4^4 = 0. \tag{8}$$

As all the non-diagonal components of the Ricci tensor R_ν^μ vanish for the line element (7), equations (2) reduce to

$$(R_1^1)^2 = (R_2^2)^2 = (R_3^3)^2 = (R_4^4)^2. \tag{9}$$

It is easily seen from equations (8) and (9) that the diagonal components of the Ricci tensor are all of the same magnitude with a pair of opposite signs, and hence there arise three possible cases:

case (i) $\qquad R_1^1 = R_2^2 = -R_3^3 = -R_4^4 \qquad (10)$

case (ii) $\qquad R_1^1 = -R_2^2 = R_3^3 = -R_4^4 \qquad (11)$

case (iii) $\qquad R_1^1 = -R_2^2 = -R_3^3 = R_4^4. \qquad (12)$

With the line element (7) the equation

$$R_1^1 = R_2^2 \tag{13}$$

is satisfied identically, and hence the last two possibilities lead to trivial cases. Thus the Rainich equations yield case (i) as the only admissible case, and we have at our disposal in case (i) two distinct equations

$$R_3^3 = R_4^4 \tag{14}$$

and

$$R_1^1 = -R_3^3 \tag{15}$$

to determine μ and ν.

With the line element (7), equation (14) reduces to

$$\ddot{\alpha} + \tfrac{1}{2}\dot{\alpha}^2 - \tfrac{1}{2}\dot{\alpha}\dot{\beta} = 0 \tag{16}$$

where we have set

$$\left.\begin{aligned} \alpha &\equiv \ln \mu \\ \beta &\equiv \ln \nu \end{aligned}\right\} \tag{17}$$

and where dots indicate differentiation with respect to time.

Likewise, equation (15) gives in view of (17)

$$3\ddot{\alpha} + 2\dot{\alpha}^2 + \ddot{\beta} + \tfrac{1}{2}\dot{\beta}^2 + \tfrac{1}{2}\dot{\alpha}\dot{\beta} = 0. \tag{18}$$

3. The solution of the problem

It is easily seen that the vector α_β as defined by (5) vanishes identically in this case, so that equations (4) are automatically satisfied. Thus our problem reduces to solving equations (16) and (18) subject to the condition (3). Equation (16) gives

$$\dot{\beta} = \frac{2\ddot{\alpha}}{\dot{\alpha}} + \dot{\alpha} \tag{19}$$

which further gives on integration with respect to time

$$\beta = 2 \ln \dot{\alpha} + \alpha + \ln A \tag{20}$$

where A is an integration constant.

Eliminating β between equations (18) and (20), we obtain

$$2\dddot{\alpha} + 7\dot{\alpha}\ddot{\alpha} + 3\dot{\alpha}^3 = 0 \tag{21}$$

which reduces to

$$2\ddot{y} + 7y\dot{y} + 3y^3 = 0 \tag{22}$$

where

$$y = \dot{\alpha}. \tag{23}$$

Equation (22) can be put in the form

$$2(\ddot{y} + ay\dot{y}) + y(b\dot{y} + 3y^2) = 0 \tag{24}$$

where a and b are constants connected by two relations, one of which is

$$2a + b = 7. \tag{25}$$

Equation (24) is satisfied if the equations

$$\ddot{y} + ay\dot{y} = 0 \tag{26}$$

and

$$b\dot{y} + 3y^2 = 0 \tag{27}$$

are satisfied simultaneously. This requirement yields

$$ab = 6 \tag{28}$$

which is the second relation between a and b. As this involves the vanishing of the integration constant obtained by integrating equation (26) once with respect to time, we shall in our latter consideration use, without loss of generality, equation (27) only in solving for y. The equations (25) and (28) give either

$$a = 2, \qquad b = 3 \tag{29}$$

or else

$$a = \tfrac{3}{2}, \qquad b = 4. \tag{30}$$

Case I

If $a = 2$ and $b = 3$, equation (27) reduces to

$$\dot{y} + y^2 = 0 \tag{31}$$

which gives on integration with respect to time

$$y = (x^4 + C)^{-1} \tag{32}$$

where C is an arbitrary constant.

On further integration, we obtain, in view of (17) and (23),

$$\mu = B(x^4 + C) \tag{33}$$

B being the constant of integration. By an obvious transformation we can write

$$\mu = Bx^4. \tag{33'}$$

Next, from equation (20) we obtain by virtue of (17), (23) and (33′)

$$\nu = \frac{AB}{x^4}. \tag{34}$$

The line element (7) thus takes the form

$$ds^2 = (dx^4)^2 - Bx^4\{(dx^1)^2 + (dx^2)^2\} - \frac{AB}{x^4}(dx^3)^2. \tag{35}$$

Here A and B are both real and $A \neq 0$, $B \neq 0$ and $AB > 0$. Also the condition (3) is satisfied for the line element (35), except when $x^4 = 0$. There appears a singularity at the origin, and as x^4 passes through the origin all the spatial components of the metric tensor (35) change sign. One may thus say that the solution exists only for $x^4 > 0$ (except for $x^4 \to \infty$) in the sense of admitting a physical interpretation.

Case II

Next we consider the case where $a = \frac{3}{2}$ and $b = 4$, and obtain as before

$$\left.\begin{aligned} \mu &= B_1(x^4)^{4/3} \\ \nu &= \frac{16AB_1}{9(x^4)^{2/3}} \end{aligned}\right\} \tag{36}$$

where B_1 is an arbitrary constant of integration.

With the metric (36)

$$R_4^4 = 0 \tag{37}$$

and consequently this leads to empty flat space.

4. The electromagnetic field

It is evident that there exists a solution for the case when the electromagnetic field is non-static and the space-time metric admits a group G_4 of automorphisms. Also, in the case under discussion, all the electromagnetic tensor components except F_{34} and F_{12} must vanish. As the electromagnetic field tensor is determined uniquely except for a duality rotation (Misner and Wheeler 1957) by the Ricci tensor and the metric tensor, one may consider the field to be either purely electric or purely magnetic or a combination of both. Subject to this uncertainty, we can in the case under consideration restrict ourselves to the purely electrostatic case when all components of the electromagnetic field tensor $F_{\mu\nu}$, except F_{34}, vanish. We then obtain for the metric (35)

$$F_{34} = \left(\frac{AB}{16\pi}\right)^{1/2} (x^4)^{-3/2} \tag{38}$$

which, in the terminology of flat space, would correspond to a component of the electric field in the x^3 direction.

It is obvious that the electromagnetic field decreases with time and vanishes for large values, as is expected for the metric (35), and one in the latter case obtains the solution for a completely empty flat space. It may be noted in passing that the electric field (38) is in fact pure imaginary if $x^4 < 0$ and is singular at $x^4 = 0$. As x^4 passes through the origin all the spatial components of the metric tensor (35) change sign, and it may be concluded that the field exists only for $x^4 > 0$, except for $x^4 \to \infty$.

It is evident from (38) that the presence of an electromagnetic field necessitates the non-vanishing of both A and B and

$$AB > 0 \tag{39}$$

which is compatible with our previous consideration.

As the metric (35) satisfies the Rainich equations and is regular for $x^4 > 0$ (except for $x^4 \to \infty$), there exists a divergence-free electromagnetic field, which could appear to be due to some phantom charges which might have pervaded all space at the singular epochs.

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